Trans-Planckian footprints in inflationary cosmology

G.L. Alberghi,* R. Casadio,† and A. Tronconi‡

Dipartimento di Fisica, Università di Bologna, and I.N.F.N,
Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy.

We consider a minimum uncertainty vacuum choice at a fixed energy scale Λ as an effective description of trans-Planckian physics, and discuss its implications for the linear perturbations of a massless scalar field in power-law inflationary models. We find possible effects with a magnitude of order H/Λ in the power spectrum, in analogy with previous results for de-Sitter space-time.

I. INTRODUCTION

Inflation has nowadays become a standard ingredient for the description of the early Universe (see, e.g., Refs. [1]). In fact, it solves some of the problems of the standard big-bang scenario and also makes predictions about cosmic microwave background radiation (CMBR) anisotropies which are being measured with higher and higher precision. Further, it has been recently suggested that inflation might provide a window towards *trans*-Planckian physics [2] (for a partial list of subsequent works on this subject, see Refs. [3, 4, 5, 6, 7, 8]). The reason for this is that inflation magnifies all quantum fluctuations and, therefore, red-shifts originally trans-Planckian frequencies down to the range of low energy physics. This causes two main concerns: first of all, there is currently no universally accepted (if at all) theory of quantum gravity which allows us to describe the original quantum fluctuations in such an high energy regime; further, it is not clear whether the red-shifted trans-Planckian frequencies can indeed be observed with the precision of present and future experiments.

Regarding the first problem, one can take the pragmatic approach of modern renormalization theory and assume that quantum fluctuations are effectively described by quantum field theory after they have been red-shifted below the scale of quantum gravity, henceforth called Λ, and forget about their previous dynamics. Further, one can also take Λ as a constant throughout the evolution of the (homogeneous and isotropic) Universe, thus implicitly assuming the existence of some preferred reference frame (class of "cosmological" observers). The second problem is instead more of a phenomenological interest and needs actual investigation to find the size of corrections to the CMBR. It then seems that the answer depends on the details of the model that one considers and no general consensus has been reached so far. In fact, in Refs. [3, 4] it is claimed that such corrections can be at most of order $(H/\Lambda)^2$, where H is the Hubble parame-

ter, hence too small to be detected. However, corrections are estimated of order H/Λ in Refs. [5, 6, 7]. Let us note that the first problem also plays an important role in this phenomenological respect, since it is the unknown trans-Planckian physics which fixes the "initial conditions" for the effective field theory description.

In Ref. [7], a principle of least uncertainty on the quantum fluctuations at the time of emergence from the Planckian domain (when the physical momentum $p \sim \Lambda$) was imposed. Without a good understanding of physics at the Planck scale, this can be regarded as an empirical way of accounting for new physics. Such a prescription fixes the initial vacuum (independently) for all frequency modes, and subsequent evolution is then obtained in the *sub*-Planckian domain by means of standard Bogolubov transformations (of course, neglecting the back-reaction) in de-Sitter space-time. In the present paper, we apply the same approach as in Ref. [7] to power-law inflation. This will allow us to check the final result against an inflationary model with time-dependent Hubble parameter.

II. SUB-PLANCKIAN EFFECTIVE THEORY

On the homogeneous and isotropic background

$$ds^2 = a^2(\eta) \left[-d\eta^2 + dx^2 + dy^2 + dz^2 \right] , \qquad (1)$$

the spatial Fourier components of the (rescaled) scalar field $\mu = a\,\phi$ (as well as tensor perturbations μ_T) satisfy

$$\mu_k'' + \left(k^2 - \frac{a''}{a} \right) \mu_k = 0 , \qquad (2)$$

where primes denote derivative with respect to the conformal time $-\infty < \eta < 0$.

The index k is related to the physical momentum p by $k = a\,p$. Thus, a given mode with energy above the Planck scale in the far past would cross the fundamental scale Λ at the time η_k when

$$k = a(\eta_k)\,\Lambda . \qquad (3)$$

Strictly speaking, it is incorrect to regard such a mode as existing for $\eta < \eta_k$, since we do not have a theory for that case. What we will in fact consider is just the evolution for $\eta > \eta_k$.

*Electronic address: alberghi@bo.infn.it
†Electronic address: casadio@bo.infn.it
‡Electronic address: tronconi@bo.infn.it

A. Minimum uncertainty principle

Following Ref. [7], we shall impose that the mode k is put into being with minimum uncertainty at $\eta = \eta_k$, that is the vacuum satisfies in the Heisenberg picture (for the details see, e.g., Ref. [9])

$$\hat{\pi}_k(\eta_k)|\,0\,\rangle = i\,k\,\hat{\mu}_k(\eta_k)|\,0\,\rangle \ , \tag{4}$$

where

$$\pi_k = \mu_k' - \frac{a'}{a}\,\mu_k \tag{5}$$

is the Fourier component of the momentum π conjugate to μ. We can write the scalar field and momentum at all times in terms of annihilation and creation operators for time dependent oscillators

$$\hat{\mu}_k(\eta) = \frac{1}{\sqrt{2k}}\left[\hat{a}_k(\eta) + \hat{a}_{-k}^\dagger(\eta)\right]$$

$$\hat{\pi}_k(\eta) = -i\sqrt{\frac{k}{2}}\left[\hat{a}_k(\eta) - \hat{a}_{-k}^\dagger(\eta)\right] \ . \tag{6}$$

The oscillators can be expressed in terms of their values at the time η_k through a Bogoliubov transformation

$$\hat{a}_k(\eta) = u_k(\eta)\,\hat{a}_k(\eta_k) + v_k(\eta)\,\hat{a}_{-k}^\dagger(\eta_k)$$

$$\hat{a}_{-k}^\dagger(\eta) = u_k^*(\eta)\,\hat{a}_{-k}^\dagger(\eta_k) + v_k^*(\eta)\,\hat{a}_k(\eta_k) \ . \tag{7}$$

Substituting this expression in (6) we obtain

$$\hat{\mu}_k(\eta) = f_k(\eta)\,\hat{a}_k(\eta_k) + f_k^*(\eta)\,\hat{a}_{-k}^\dagger(\eta_k)$$

$$i\,\hat{\pi}_k(\eta) = g_k(\eta)\,\hat{a}_k(\eta_k) - g_k^*(\eta)\,\hat{a}_{-k}^\dagger(\eta_k) \ . \tag{8}$$

where

$$f_k(\eta) = \frac{1}{\sqrt{2k}}\left[u_k(\eta) + v_k^*(\eta)\right]$$

$$g_k(\eta) = \sqrt{\frac{k}{2}}\left[u_k(\eta) - v_k^*(\eta)\right] \ , \tag{9}$$

and $f_k(\eta)$ is a solution of the mode equation (2). The condition (4) then reads

$$v_k(\eta_k) = \sqrt{\frac{k}{2}}\,f_k^*(\eta_k) - \frac{1}{\sqrt{2\,k}}\,g_k^*(\eta_k) = 0 \ . \tag{10}$$

This requirement, together with the normalization condition

$$|u_k|^2 - |v_k|^2 = 1 \ , \tag{11}$$

is sufficient to determine uniquely the initial state at $\eta = \eta_k$. The subsequent time evolution is then straightforward and one can estimate the power spectrum of fluctuations at a later time $\eta \gg \eta_k$ after the end of inflation,

$$P_\phi = \frac{P_\mu}{a^2} = \frac{k^3}{2\,\pi^2\,a^2}\,|f_k(\eta)|^2 \ . \tag{12}$$

The above general formalism was applied to de-Sitter space-time in Ref. [7]. For that case, one has $a = -1/H\,\eta$ and the nice feature follows that

$$k\,\eta_k = -\frac{\Lambda}{H} \tag{13}$$

is a constant independent of k. This, in turn, allows to obtain an analytic expression for the initial state which satisfies Eq. (10) by suitably expanding for H/Λ small (i.e., $\eta_k \to -\infty$ for all k). We shall instead consider power-law inflation, where such a simplification does not occur.

B. Power-law inflation

In the proper time $dt = a\,d\eta$, power-law inflation is given by a scale factor $a \sim t^p$, in which $t_{\rm p} < t < t_{\rm o}$, with $t_{\rm p}$ of the order of the Planck time, $t_{\rm o} \gg t_{\rm p}$ is the time of the end of inflation, and $p \gg 1$ [5]. Upon changing to the conformal time, one obtains for the scale factor

$$a(\eta) = \left(\frac{\bar{\eta}}{\eta}\right)^q \ , \tag{14}$$

where $q = p/(p-1)$, $\eta_{\rm p} < \eta \le \eta_{\rm o} < 0$ ($\eta_{\rm o}$ is the end of inflation) and the Hubble parameter is given by

$$H(\eta) = -q\,\frac{\eta^{q-1}}{\bar{\eta}^q} \ . \tag{15}$$

The condition (3) now becomes

$$k\,\eta_k = \bar{\eta}\,\Lambda^{\frac{1}{q}}\,k^{1-\frac{1}{q}} \ . \tag{16}$$

Since the right hand side depends on k (unless $q = 1$), it can be large or small depending on k, and an expansion for $-k\,\eta_k$ large is not generally valid.

For the scale factor (14) one has

$$\frac{a''}{a} = \frac{q\,(q+1)}{\eta^2} \ , \tag{17}$$

and Eq. (2) can be solved exactly. One can write the general solution as

$$f_k = A_k\,\sqrt{-\eta}\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,\sqrt{-\eta}\,Y_{q+\frac{1}{2}}(-k\,\eta) \ , \tag{18}$$

where J_ν and Y_ν are Bessel functions of the first and second kind [12], and A_k and B_k are complex constants. The Bogolubov coefficients are then given by

$$u_k = \sqrt{-\frac{k\,\eta}{2}}\left[A_k\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q+\frac{1}{2}}(-k\,\eta)\right.$$
$$\left. -i\left(A_k\,J_{q-\frac{1}{2}}(-k\,\eta) + B_k\,J_{q-\frac{1}{2}}(-k\,\eta)\right)\right] \tag{19}$$

$$v_k^* = \sqrt{-\frac{k\,\eta}{2}}\left[A_k\,J_{q+\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q+\frac{1}{2}}(-k\,\eta)\right.$$
$$\left. +i\left(A_k\,J_{q-\frac{1}{2}}(-k\,\eta) + B_k\,Y_{q-\frac{1}{2}}(-k\,\eta)\right)\right] \ .$$

The constants A_k and B_k can now be fixed by imposing the normalization condition (11) and Eq. (10). From Eq. (11) one obtains

$$A_k B_k^* - A_k^* B_k = -i\frac{\pi}{2} \; , \qquad (20)$$

and from Eq. (10),

$$A_k = -\frac{\bar{Y}_{q+\frac{1}{2}} + i\,\bar{Y}_{q-\frac{1}{2}}}{\bar{J}_{q+\frac{1}{2}} + i\,\bar{J}_{q-\frac{1}{2}}}\,B_k \; , \qquad (21)$$

where $\bar{J}_\nu \equiv J_\nu(-k\,\eta_k)$ and $\bar{Y}_\nu \equiv Y_\nu(-k\,\eta_k)$. From the combined equations one then obtains

$$|A_k|^2 = -\frac{\pi^2}{8}\,k\,\eta_k\,\left[\bar{Y}_{q+\frac{1}{2}}^2 + \bar{Y}_{q-\frac{1}{2}}^2\right]$$

$$|B_k|^2 = -\frac{\pi^2}{8}\,k\,\eta_k\,\left[\bar{J}_{q+\frac{1}{2}}^2 + \bar{J}_{q-\frac{1}{2}}^2\right] \qquad (22)$$

$$\mathrm{Re}\,(A_k B_k^*) = \frac{\pi^2}{8}\,k\,\eta_k\,\left(\bar{Y}_{q+\frac{1}{2}}\bar{J}_{q+\frac{1}{2}} + \bar{Y}_{q-\frac{1}{2}}\bar{J}_{q-\frac{1}{2}}\right) \; .$$

We are finally in the position to compute the exact power spectrum at the time $\eta \le \eta_\mathrm{o}$, which is given by

$$P_\phi = \frac{\eta_k\,\eta^{2\,q+1}\,k^4}{16\,\bar{\eta}^{2\,q}}\left\{\left[\bar{Y}_{q+\frac{1}{2}}\,J_{q+\frac{1}{2}}(-k\,\eta) - \bar{J}_{q+\frac{1}{2}}\,Y_{q+\frac{1}{2}}(-k\,\eta)\right]^2 + \left[\bar{Y}_{q-\frac{1}{2}}\,J_{q+\frac{1}{2}}(-k\,\eta) - \bar{J}_{q-\frac{1}{2}}\,Y_{q+\frac{1}{2}}(-k\,\eta)\right]^2\right\} \; . \qquad (23)$$

The above expression can then be estimated for $\eta = \eta_\mathrm{o}$ (end of inflation) and $\eta_\mathrm{o} \to 0^-$. Since for $-k\,\eta_\mathrm{o} \ll 1$, the Bessel $Y_{q+\frac{1}{2}}$ dominates, one obtains, to leading order,

$$P_\phi \simeq \frac{k^{3-2\,q}\,|\eta_k|}{2^{3-2\,q}\,|\bar{\eta}|^{2\,q}}\,\frac{\bar{J}_{q+\frac{1}{2}}^2 + \bar{J}_{q-\frac{1}{2}}^2}{\sin^2\left(\pi\,(q+\frac{1}{2})\right)\,\Gamma^2\left(\frac{1}{2}-q\right)} \; . \qquad (24)$$

If one further takes the limit $k\,\eta_k \to -\infty$ and expands to leading order for k small, the power spectrum becomes

$$
\begin{aligned}
P_\phi \;\simeq\; & \frac{2^{2\,q-2}\,k^{2-2\,q}}{\pi\,|\bar{\eta}|^{2\,q}\,\cos^2(\pi\,q)\,\Gamma^2\left(\frac{1}{2}-q\right)} \\
& \times \left[1 - \frac{H_k}{\Lambda}\,\sin(2\,\bar{\eta}\,\Lambda^{\frac{1}{q}}\,k^{1-\frac{1}{q}} + \pi\,q)\right] \\
=\; & P_\mathrm{PL}\left[1 - \frac{H_k}{\Lambda}\,\sin\left(q\,\frac{2\,\Lambda}{H_k} + q\,\pi\right)\right] \; , \qquad (25)
\end{aligned}
$$

where $H_k \equiv H(\eta_k)$ and we have factored out the expression $P_\mathrm{PL} \sim k^{2-2\,q}$ of the spectrum for power-law inflation [10] in the small $k\,\eta_\mathrm{o}$ regime (super-horizon scales) [11]. This result is thus in agreement with what was obtained for de-Sitter space-time in Ref. [7], as one can easily see by taking the limit $q \to 1$ ($p \to \infty$).

However, as we mentioned previously, $k\,\eta_k$ is not independent of k [see Eq. (16)]. The above expression therefore does not hold for all k, but just for those such that $-k\,\eta_k$ is large. Since it is very difficult to obtain general analytic estimates of the exact power spectrum for general values of k, in Fig. 1 we plot, for the exact expression of P_ϕ in Eq. (23), the ratio

$$R_q = \frac{P_\phi - P_\mathrm{PL}}{P_\mathrm{PL}} \; , \qquad (26)$$

for $q = 2$, $3/2$ and $4/3$ (similar results are obtained for all values of $q \ne 1$). It is clear that for small k the oscillations in P_ϕ are relatively large around P_PL, and this is precisely due to the dependence of $k\,\eta_k$ on k. The oscillations are then progressively damped for large k according to the approximate expression in Eq. (25) (and analogously to what is found in de-Sitter [7]). Note also that for increasing p (i.e. $q \to 1^+$), the wavelength of oscillations increases, as is shown in the approximation (25). Of course, one must keep in mind that only sub-horizon scales matter at the time η_k, for which $k \gg a\,H$, that is $|k\,\eta_k| \gg q$ (say of order λ). Hence, the relevant regions for different values of q are those with $k \gtrsim \lambda^{q/(q-1)}$. In Fig. 1 we have set $\lambda = 10$ in order to obtain reasonably overlapping ranges, and the amplitude of the oscillations turns out to be of the order of a few percents inside the physical ranges (larger values of λ imply smaller oscillations).

III. CONCLUSIONS

We considered a minimum uncertainty principle to fix, at an energy scale Λ, the vacuum of an effective (low energy) field theory. Such prescription involves the cut-off scale Λ for dealing with trans-Planckian energies, which therefore enters into the power spectrum of perturbations at later times. We have shown in some details that a Λ of the order of the Planck scale can affect appreciably the spectrum [see Eq. (25) and Fig. 1], in agreement with Refs. [5, 6, 7] by introducing a modulation of the spectrum, as may be clearly seen from the figure. This is a clear indication that trans-Planckian physics can lead to observable predictions in the cosmological models. We feel this is further evidence for the fact that trans-Planckian physics cannot be safely ignored in determining observable quantities such as features of the CMBR.
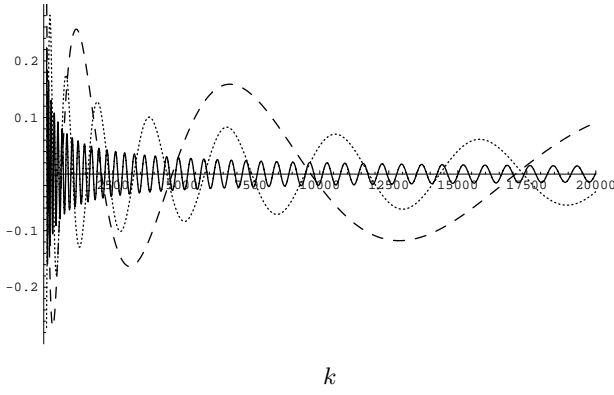
FIG. 1: The ratio R_q for P_ϕ in Eq. (23) and $q = 2$ (solid line), $q = 3/2$ (dotted line) and $q = 4/3$ (dashed line). The momentum index k is in units with $\Lambda = \bar{\eta} = 1$ and the regions of physical interest are those for $k \gtrsim 10^2$ ($q = 2$), $k \gtrsim 10^3$ ($q = 3/2$) and $k \gtrsim 10^4$ ($q = 4/3$).

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